Math 3063 Abstract Algebra Project 4 Solutions
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Copy the statement of the problem on a piece of $8\frac{1}{2} \times 11$ piece of blank computer paper, and write the solution underneath. Write neatly. Mathematics should always be written in grammatically correct English, in complete sentences.

If $G$ is a group, $H$ is a subgroup of $G$, and $K$ is a subgroup of $H$, then $K$ is a subgroup of $G$.

If $G$ is a group, and $H$ and $K$ are subgroups of $G$, then their intersection $H \cap K$ is a subgroup of $G$.

A permutation $\alpha \in S_n$ is called *even* if it can be written as a product of an even number of transpositions; otherwise it is called *odd*. Exactly half of the permutations in $S_n$ are even.

Set
$$A_n = \{\alpha \in S_n \mid \alpha \text{ is even}\}.$$
Then $A_n$ is a subgroup of $S_n$, called the *alternating subgroup*.

Let $H$ be a subgroup of $S_n$. Then either $H$ consists of even permutations or exactly half of the permutations in $H$ are even. Thus either $H \subset A_n$, in which case $H \cap A_n = H$, or $H \cap A_n$ is exactly half of $H$. We outline the proof. Suppose that $H$ is not contained in $A_n$ and let $K = H \cap A_n$; we want to show that $|H| = 2|K|$. Let $\alpha \in H$ be an odd permutation. Set $\alpha K = \{\alpha\kappa \mid \kappa \in K\}$. Then $K \cup \alpha K = H$, $K \cap \alpha K = \varnothing$, and $|K| = |\alpha K|$.

Let $\rho, \tau \in S_n$ be given by
$$\rho = \texttt{(1 2 ... n)} \quad \text{and} \quad \tau = \begin{cases} \texttt{(2 n)(3 n-1) ... ((n+1)/2 (n+3)/2)} & \text{if } n \text{ is odd;} \\ \texttt{(2 n)(3 n-1) ... (n/2 (n+4)/2)} & \text{if } n \text{ is even.} \end{cases}$$

Set
$$D_n = \{\epsilon, \rho, \rho^2, \dots, \rho^{n-1}, \tau, \tau\rho, \tau\rho^2, \dots, \tau\rho^{n-1}\} \subset S_n.$$
Then $D_n$ is a subgroup of $S_n$, called the *dihedral subgroup*. The proof that this is a subgroup follows from the identity $\tau\rho = \rho^{n-1}\tau$.

Set $K_n = D_n \cap A_n$. Then $K_n$ is a subgroup of $S_n$, and either $K_n = D_n$ or $K_n$ is exactly half of $D_n$. This quiz examines the relationship between $n$ and the structure of the group $K_n$.

**Problem 1.** Let $n = 4$.
**(a)** Compute $\rho$ and $\tau$ in this case.
**(b)** Show that $K_4$ is a noncyclic abelian subgroup of $S_4$.

*Solution.* Let
$$\rho = \texttt{(1 2 3 4)} \text{ and } \tau = \texttt{(2,4)}.$$
Then
$$K_4 = \{\epsilon, \rho^2, \tau\rho, \tau\rho^3\} = \{\epsilon, \texttt{(1,2)(3,4)}, \texttt{(1,3)(2,4)}, \texttt{(1,4)(2,3)}\}.$$
Since $|K_4| = 4$ and $K_4$ does not contain an elements of order four, it is not cyclic. Since every element in $K_4$ has order two, it is abelian. □

**Problem 2.** Let $n = 5$.
**(a)** Compute $\rho$ and $\tau$ in this case.
**(b)** Show that $K_5 = D_5$.

*Solution.* Let
$$\rho = \texttt{(1 2 3 4 5)} \text{ and } \tau = \texttt{(2,5)(3,4)}.$$
Then $D_5 = \langle \rho, \tau \rangle$. Since every nontrivial rotation in $D_5$ is a five cycle, they are all even, and since every reflection fixes one point, they are pairs of transpositions, and so are also even. Thus $D_5 \leq A_5$, so $K_5 = D_5 \cap A_5 = D_5$. □

**Problem 3.** Let $n = 7$.
**(a)** Compute $\rho$ and $\tau$ in this case.
**(b)** Show that $K_7$ is a cyclic subgroup of $S_7$.

*Solution.* Let

$$\rho = \texttt{(1 2 3 4 5 6 7)} \text{ and } \tau = \texttt{(2,7)(3,6)(4,5)}.$$

Every rotation is even, and every reflection is odd. Thus $K_7 = \langle \rho \rangle = C_7$. □

**Problem 4.** Try to generalize the previous problems: what can you say about $K_n$ in the following cases?
**(a)** $n \equiv 0 \pmod 4$
**(b)** $n \equiv 1 \pmod 4$
**(c)** $n \equiv 2 \pmod 4$
**(d)** $n \equiv 3 \pmod 4$

*Proof.* If $n$ is odd, then $\rho$ is a cycle of odd length, so $\rho$ is an even permutation. In this case, every power of $\rho$ is also even, so

$$\langle \rho \rangle = C_n \leq A_n.$$

Since $C_n$ is exactly half of $D_n$, if one reflection is even, then they all are, because $K_n$ is either exactly half of $D_n$, or it is all of $D_n$.

If $n$ is even, then $\rho$ is a cycle of even length, so $\rho$ is an odd permutation. In this case, even powers of $\rho$ are even permutation, and odd powers of $\rho$ are odd permutations. Thus

$$C_n \cap A_n = \langle \rho^2 \rangle.$$

Now $\tau$ always fixes 1, and when $n$ is even, $\tau$ fixes two points. The support of $\tau$ contains $n - 2$ points, so $\tau$ consists of $(n - 2)/2$ disjoint transpositions. If $n \equiv 0 \pmod 4$, then $(n - 2)/2$ is odd, so $\tau\rho$ is even. If $n \equiv 2 \pmod 4$, then $(n - 2)/2$ is even, but $\tau\rho$ is odd. In either case, exactly half of the reflections are even, and $|K_n| = n/2$.

**(b)** $n \equiv 1 \pmod 4$
We have $C_n \leq A_n$, so $C_n \leq K_n$. Every reflection fixes exactly one point, so its support has size $n - 1$, which is divisible by 4. The number of transpositions in a reflection is $\frac{n-1}{2}$, which is still even, so all the reflections are in $A_n$. Thus $K_n = D_n$.

**(d)** $n \equiv 3 \pmod 4$
We have $C_n \leq A_n$, so $C_n \leq K_n$. Every reflection fixes exactly one point, so its support has size $n - 1$, which is divisible by 2 but not 4. The number of transpositions in a reflection is $\frac{n-1}{2}$, which is odd in this case, so none of the reflections are in $A_n$. Thus $K_n = C_n$.

**(c)** $n \equiv 2 \pmod 4$
We know that $K_n$ contains a cyclic subgroup of order $\frac{n}{2}$ generated by $\rho^2$. Indeed, $K_n$ consists of even powers of $\rho$, and elements of the form $\tau\rho^k$ where $k$ is even.

Let $m = \frac{n}{2}$, and define a function $\phi : K_n \to D_m$ by $\phi(\alpha) = \phi_\alpha$, where $\phi_\alpha \in S_m$ is given by $\phi_\alpha(i) = \frac{\alpha(i)-1}{2} + 1$. Then $\phi$ is an isomorphism. □

**(d)** $n \equiv 0 \pmod 4$ Here again, we have $K_n \cong D_{n/2}$, but the correspondence is a little harder to see because the reflections which allow for visualization of the correspondence have no fixed points.

Let $m = \frac{n}{2}$, and define a function $\phi : K_n \to D_m$ by $\phi(\alpha) = \phi_\alpha$, where $\phi_\alpha \in S_m$ is given by $\phi_\alpha(i) = i \pmod 3$. Then $\phi$ is an isomorphism.